

Vanishing and non-vanishing of Ceresa cycles

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Outline

Let C be an algebraic curve embedded in its Jacobian $J = H^0(C, \Omega_C)^* / H_1(C, \mathbb{Z})$.

Question: Does C “algebraically deform” to $-C$? More precisely, for which curves C does the Ceresa cycle $[C] - [-C]$ vanish modulo algebraic equivalence?

Interesting from both a geometric and arithmetic point of view (see Wei Zhang’s talk for applications to L -functions).

I’ll discuss geometric and arithmetic methods to study this question.

- Vanishing results with Jef Laga (2025)
- Non-vanishing results with Naomi Sweeting (forthcoming)

Algebraic cycles and equivalence

X smooth, projective algebraic variety over a field k .

$Z_1(X)$ = free abelian group on closed (integral) curves on X .

Have various equivalence relations on 1-cycles:

$$Z_1(X)_{\text{rat}} \subset Z_1(X)_{\text{alg}} \subset Z_1(X)_{\text{hom}} \subset Z_1(X).$$

- Two 1-cycles are **homologically equivalent** if they agree in cohomology.
- If $S \subset X$ is a surface and $f: S \rightarrow C$ is a map to a curve, then the 1-cycles $f^{-1}(c)$ are **algebraically equivalent** to each other.
- **Rational equivalence** is defined as above, but allowing only $C = \mathbb{P}^1$.

Griffiths groups

Taking quotients modulo rational equivalence gives:

$$0 \subset \mathrm{CH}_1(X)_{\mathrm{alg}} \subset \mathrm{CH}_1(X)_{\mathrm{hom}} \subset \mathrm{CH}_1(X)$$

- If $\dim X \leq 2$ then $\mathrm{CH}_1(X)_{\mathrm{alg}} = \mathrm{CH}_1(X)_{\mathrm{hom}}$
- Griffiths: if $\dim X \geq 3$, then the group

$$\mathrm{Gr}_1(X) := \mathrm{CH}_1(X)_{\mathrm{hom}} / \mathrm{CH}_1(X)_{\mathrm{alg}}$$

need not be 0!

- Example: difference $[L] - [L']$ of lines on general quintic threefold $X \subset \mathbb{P}^4$.

Ceresa cycles

- Let C be a curve of genus $g \geq 2$.
- Let $J = \text{Pic}^0(C)$ be its Jacobian, a g -dimensional abelian variety.
- Fix $e = \frac{1}{2g-2}K_C \in \text{Pic}^1(C)$ and embed $C \xrightarrow{\iota} J$ via $x \mapsto x - e$.
- Define $\kappa(C) = [\iota(C)] - [-\iota(C)] \in \text{CH}_1(J)$.

Theorem (Ceresa '83)

For very general C over \mathbb{C} of genus $g \geq 3$, $\kappa(C)$ has infinite order in $\text{Gr}_1(J)$.

Main tool: Higher Abel-Jacobi map

$$\text{CH}_1(J)_{\text{hom}} \xrightarrow{\text{AJ}} J_1(J) := \frac{\text{Fil}^2 \text{H}^3(J, \mathbb{C})^*}{\text{H}_3(J, \mathbb{Z})}$$
$$\kappa \quad \mapsto \quad \left[\omega \mapsto \int_{\partial^{-1}\kappa} \omega \right]$$

Modified diagonal cycles

We can alternatively work with the triple product C^3 instead of J .

Let $\Delta_{123} = \{(x, x, x): x \in C\}$, let $\Delta_{12} = \{(x, x, e): x \in C\}$, etc.

The **Gross-Schoen modified diagonal cycle** is

$$\Delta(C) = \Delta_{123} - \Delta_{12} - \Delta_{13} - \Delta_{23} + \Delta_1 + \Delta_2 + \Delta_3 \in \text{CH}_1(C^3)$$

Theorem (Zhang)

$\kappa(C)$ is infinite order in $\text{Gr}_1(J)$ if and only if $\Delta(C)$ is infinite order in $\text{Gr}_1(C^3)$.

Clemens' question

“Most” curves have infinite order Ceresa cycle, but **which ones?**

Example: If C is hyperelliptic then $\kappa(C) = 0$ in $\text{Gr}_1(J)$.

Question (Clemens '87)

$\kappa(C) = 0$ in $\text{Gr}_1(J)$ if and only if C is hyperelliptic?

Variants: When does $\kappa(C)$ vanish in the Chow group? When is it torsion?

In all variants, the spirit is:

The Ceresa class vanishes/is torsion only if there is a good reason.

Possible counterexamples to Clemens' question?

Examples were recently found with $\text{AJ}(\kappa(C))$ torsion, exploiting $G = \text{Aut}(C)$.

- Fricke-Macbeath curve: $g = 7$ and $G \simeq \text{PSL}_2(\mathbb{F}_8)$. (Bisogno-Li-Litt-Srinivasan)
- The genus 3 curve $C : y^3 = x^4 + x$ with $G \simeq C_9$. (Beauville)

Proof.

AJ is G -equivariant and $\kappa(C)$ is canonical and hence G -invariant. One computes that the target $J^{g-1}(J)^G \otimes \mathbb{Q} = 0$, hence $\text{AJ}(\kappa(C))$ is torsion. \square

Beilinson-Bloch conjecture: the ℓ -adic AJ map

$$\text{CH}_1(J)_{\text{hom}} \otimes \mathbb{Q}_\ell \rightarrow H^1(\text{Gal}_{\mathbb{Q}}, H^{2g-3}(J_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(g-1)))$$

is an injection. This implies $\kappa(C)$ is torsion. Can we prove it unconditionally?

Chow-vanishing criterion

Let $H^*(X) = H^*(X(\mathbb{C}), \mathbb{Q})$. So $H^3(J) = \wedge^3 H^1(C)$.

Theorem (Laga-S)

If $H^3(J)^G = 0$ then $\kappa(C)$ is torsion in $\text{CH}_1(J)$.

- Strengthens a result of Qiu-Zhang.

Examples:

- $y^3 = x^4 + 1$ (Qiu-Zhang, $g = 3$, $G = \text{SmallGroup}(48, 33)$)
- $y^3 = x^4 + x$ (Beauville-Schoen, $g = 3$, $G = C_9$)
- $y^3 = x^5 + 1$ (Lilienfeldt-S, $g = 4$, $G = C_{15}$)
- $C_t: y^3 = (x^3 + t)^2(tx^3 - 1)$ (Qiu-Zhang, $g = 4$, $G = S_3^2$)
- Humbert-Edge curves (Laterveer, $g = 5$, $G = C_2^4$)
- $\kappa(C) = 0$ and $C \rightarrow D \implies \kappa(D) = 0$. E.g. Fricke-Macbeath quotient of $g = 3$.

Griffiths-vanishing criterion

For algebraic equivalence, we consider the g -dimensional G -rep $V = H^0(C, \Omega_C)$.

Theorem (Laga-S)

Assume the Hodge conjecture. If $(\wedge^3 V)^G = 0$ then $\kappa(C)$ is torsion in $\mathrm{Gr}_1(J)$.

- The hypotheses imply $H^3(J)^G \simeq H^1(A)(-1)$ for some abelian variety A .
- Need the Hodge conjecture for $J \times A$.

Examples

- The genus three family of Picard curves $y^3 = x^4 + ax^2 + bx + c$
 - $G = C_3$. HC known.
- The genus 4 family $y^5 = x(x-1)^2(x-t)^3$
 - $G = D_5$. HC known.
- The genus 4 family $y^3 = x^2(x-1)^2(x^3 + ax^2 + bx + c)$. HC?
- The genus 6 family $y^9 = x(x-1)^3(x-t)^6$. HC?
 - This covers a genus 3 family with $\text{Aut}(C) = 0$!
- Are there families of arbitrarily large genus?

Can $\kappa(C)$ vanish for non-group theoretic reasons? Yes!

For separable $f(x) = x^4 + ax^2 + bx + c$, let C_f be the genus 3 curve $y^3 = f(x)$.

Define the invariants

- $I(f) = a^2 + 12c$
- $J(f) = 72ac - 2a^3 - 27b^2$

The 19-th century invariant theorists observed: $J(f)^2 = 4I(f)^3 - 27\text{Disc}(f)$.

In other words, $P_f := (I(f), J(f))$ lies on the elliptic curve $E_f: y^2 = 4x^3 - 27\text{Disc}(f)$.

Theorem (Laga-S)

$\kappa(C_f)$ is torsion in $\text{CH}_1(J)$ if and only if P_f is a torsion point in $E_f(\mathbb{C})$.

Proof sketch

Let S be the parameter space of polynomials f .

Have a family $\mathcal{C} \rightarrow S$ of curves with fiber-wise algebraically trivial Ceresa cycle.

The holomorphic **normal function** $S \xrightarrow{\sigma} \mathcal{J}_1(\mathcal{J})$ to the family of intermediate Jacobians factors through an algebraic section of an abelian scheme $\mathcal{A} \rightarrow S$.

Check that $\mathcal{A} = E_f$ and compute that $\mathcal{A}(S) \simeq \mathbb{Z}$ via Shioda-Tate formula.

Show $\sigma \neq 0$ by specializing and verifying on a single example!

Corollaries

- Infinitely many 1-parameter families of plane quartics curves over $\overline{\mathbb{Q}}$ with $\kappa(C)$ torsion in $\text{CH}_1(J)$. For example $C_t: y^3 = x^4 - 12x^2 + tx - 12$.
 - Question: does $\kappa(C_t)$ always have order 3?
- Over $\overline{\mathbb{Q}}$, there are torsion classes $\kappa(C_f)$ of unbounded order.
- Over number fields K of fixed degree, the torsion order of $\kappa(C_f)$ is bounded.

Question

For each $N \geq 1$, is $\{C \in \mathcal{M}_g: \kappa(C) \text{ has order at most } N\}$ Zariski closed in \mathcal{M}_g ?

Question

Is there a uniform bound for the order of a torsion Ceresa cycle $\kappa(C)$ over \mathbb{Q} ?

Non-vanishing results

Theorem (Gao-Zhang, '24)

For $g \geq 3$, there is an open dense set $\mathcal{M}_g^{amp} \subset \mathcal{M}_g$ such that the Beilinson-Bloch height of $\Delta(C)$ satisfies a Northcott property on $\mathcal{M}_g^{amp}(\bar{\mathbb{Q}})$.

A very strong non-vanishing result! For example, it implies that 100% of plane quartic curves over \mathbb{Q} have $\Delta(C)$ of infinite order. **But what is the open set \mathcal{M}_g^{amp} ?!**

Known non-vanishing results for $\Delta(C)$ or $\kappa(C)$ fall into three types:

- 1 Large families but inexplicit non-vanishing
(Ceresa, Collino-Pirola, Gao-Zhang, Hain, Kerr-Tayou)
- 2 Special families (Eskandari-Murty, Kerr-Li-Qiu-Yang, Laga-S, Luopian-Rawson, Bhatnagar-Devadas-D'Nelly-Warady-Srinivasan)
- 3 Algorithms for individual curves (Bloch, Ellenberg-Logan-Srinivasan)

Non-vanishing criteria via bad reduction

Let K be a number field, ℓ a prime, and v_1, v_2 two primes of K .

In forthcoming work with Naomi Sweeting, we show that for certain fixed choices of

- 1 bad reduction type of C at v_1 and
- 2 Galois module structures $J[\ell](\overline{K}_{v_2})$ at v_2 ,

the class $\Delta(C)$ has infinite order.

Since (1) and (2) are v -**adically locally constant**, get many examples this way.

For example, we give explicit Zariski dense subsets of the $\mathcal{M}_3(\mathbb{Q})$ (\approx space of plane quartic curves) consisting of curves with infinite order $\Delta(C)$. A similar construction should work for every degree d .

The potential of $\Delta(C)$

We prove this by computing the “potential” of the Gross-Schoen cycle $\Delta(C)$.

Yifeng Liu’s **potential map** can be thought of as a homomorphism

$$\text{Pot}_v: \text{CH}_1(C^3)_{\text{hom}} \rightarrow H^1(\text{Gal}_K, H^3(C^3, \mathbb{Z}/\ell^n(2))) \rightarrow H^1(I_v, H^3(C^3, \mathbb{Z}/\ell^n(2)))^{\text{Frob}_v}.$$

The computation of $\text{Pot}_v(\Delta(C))$ is long and intricate, using the work of Gross-Schoen and the weight-monodromy spectral sequence.

We show that it recovers the tropical Ceresa class of the special fiber of C , as defined recently by Amini–Corey–Monin.

Connection with tropical Ceresa

Let C be defined over a finite extension of \mathbb{Q}_p with residue field k .

Say C has bad (strictly) semistable reduction. Let Γ be the graph whose vertices are the irreducible components of the special fiber C_k and whose edges are intersections.

Suppose $e = \sum a_i P_i \in \text{Pic}^1(C)$ with each P_i reducing to a smooth point of a component v_i on the special fiber. Let $e_0 = \sum a_i v_i$, a divisor on Γ .

Amini–Corey–Monin define a tropical Ceresa class

$$\kappa_{e_0}(\Gamma) := \sum_{\alpha \in E} a_\alpha \otimes b_\alpha \wedge (p_{s(\alpha)} + p_{t(\alpha)})$$

in the finite group $\text{coker} \left(\wedge^2 H_1(\Gamma) \otimes H^1(\Gamma) \xrightarrow{N} H_1(\Gamma) \otimes \wedge^2 H^1(\Gamma) \right)$.

Theorem (S–Sweeting)

The order of the tropical class $\kappa_{e_0}(\Gamma)$ divides the order of $\text{Pot}_v(\Delta(C))$.

Example

This result is only useful if the graph Γ is interesting enough. In particular, C should have bad reduction and Γ should not be a tree.

Example

Suppose C_k has two smooth irreducible components intersecting at three simple nodes, so that Γ is the graph $\Gamma(2,3)$ with two vertices connected by three edges. Then the order of $\kappa_{e_0}(\Gamma)$ is divisible by 3.

Theorem (S-Sweeting)

Suppose C is a curve over \mathbb{Q} with reduction of type $\Gamma(2,3)$ at some prime $p \neq 3$ and with mod 3 Galois image equal to $\mathrm{GSp}_{2g}(\mathbb{F}_3)$. Then $\Delta(C)$ has infinite order.

Applications

- 1 “Most” families of genus $g \geq 3$ algebraic curves over a subset of $\mathbb{A}_{\mathbb{Q}}^n$ will allow for reduction types Γ with non-trivial tropical Ceresa class. So this gives an approach to prove an explicit version of Ceresa’s theorem for “most” families of curves.
- 2 Verifying exact torsion orders. For example, one can show that many of the curves $C_t: y^3 = x^4 - 12x^2 + tx - 12$ from before have $\kappa(C_t)$ of order at *least* 3.
- 3 One can cook up curves of fixed genus with $\kappa_{e_0}(\Gamma)$ having arbitrarily large order. For example, the genus 3 curves

$$p^t(ax + by + cz)^4 + xyz(x + y + z) + p^s xyQ(x, y, z) = 0$$

have $\Gamma = K_4$ special fibers and arbitrarily large tropical Ceresa orders as s, t vary. Can one prove unconditionally that these curves have $\Delta(C)$ of infinite order?

